

CUBIC FOURFOLDS FIBERED IN SEXTIC DEL PEZZO SURFACES

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ABSTRACT. We exhibit new examples of rational cubic fourfolds. Our examples admit fibrations in sextic del Pezzo surfaces over the projective plane, which are rational whenever they have a section. They are parametrized by a countably infinite union of codimension-two subvarieties in the moduli space.

The rationality problem for complex cubic fourfolds has been studied by many authors; see [Has16, §1] for background and references to the extensive literature on this subject. The moduli space \mathcal{C} of smooth cubic fourfolds has dimension 20. Since the 1990's, the only cubic fourfolds *known* to be rational are:

- Pfaffian cubic fourfolds and their limits. These form a divisor $\mathcal{C}_{14} \subset \mathcal{C}$.
- Cubic fourfolds containing a plane $P \subset X$ such that the induced quadric surface fibration $\mathrm{Bl}_P(X) \rightarrow \mathbb{P}^2$ has an odd-degree multisection. These form a countably infinite union of codimension-two loci $\bigcup \mathcal{C}_K \subset \mathcal{C}$, dense in the divisor \mathcal{C}_8 parametrizing cubic fourfolds containing a plane.

Our main result is:

Theorem 1. *Let $\mathcal{C}_{18} \subset \mathcal{C}$ denote the divisor of cubic fourfolds of discriminant 18. There is a Zariski open subset of $U \subset \mathcal{C}_{18}$ and a countably infinite union of codimension-two loci $\bigcup \mathcal{C}_K \subset \mathcal{C}$, dense in \mathcal{C}_{18} , such that $\bigcup \mathcal{C}_K \cap U$ parametrizes rational cubic fourfolds.*

We show that the generic element X of \mathcal{C}_{18} is birational to a fibration in sextic del Pezzo surfaces over \mathbb{P}^2 . Its generic fiber and thus X are rational if the fibration admits a section. In fact, a multisection of degree prime to three suffices to establish rationality of X . This condition can be expressed in Hodge-theoretic terms, and is satisfied along a countably infinite union of divisors in \mathcal{C}_{18} .

In Section 1 we show that a generic cubic fourfold in \mathcal{C}_{18} contains an elliptic ruled surface. In Section 2 we construct fibrations in sextic

del Pezzo surfaces from elliptic ruled surfaces. After reviewing rationality of sextic del Pezzo surfaces in Section 3, we prove Theorem 1 in Section 4. In Section 5 we analyze the degenerate fibers of fourfolds fibered in sextic del Pezzo surfaces. Our approach is grounded in assumptions on the behavior of ‘generic’ cases; Section 6 validates these in a concrete example.

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1. SEXTIC ELLIPTIC RULED SURFACES AND CUBIC FOURFOLDS

Theorem 2. *A generic cubic $X \in \mathcal{C}_{18}$ contains an elliptic ruled surface T of degree 6, with the following property: the linear system of quadrics in \mathbb{P}^5 containing T is 2-dimensional, and its base locus is a complete intersection $\Pi_1 \cup T \cup \Pi_2$, where Π_1 and Π_2 are disjoint planes, which meet T along curves E_i that give sections of the \mathbb{P}^1 -fibration $T \rightarrow E$.*

Before proving this we prove two preliminary results. We take the planes Π_1, Π_2 as our starting point.

Proposition 3. *Let $\Pi_1, \Pi_2 \subset \mathbb{P}^5$ be two disjoint planes. Then for a generic choice of three quadrics Q_1, Q_2, Q_3 containing Π_1, Π_2 , we have*

$$Q_1 \cap Q_2 \cap Q_3 = \Pi_1 \cup T \cup \Pi_2,$$

where T is an elliptic ruled surface of degree 6. Moreover, the curves $E_i := T \cap \Pi_i$ are sections of the \mathbb{P}^1 -fibration $T \rightarrow E$.

Proof. Observe that a point in $\mathbb{P}^5 \setminus (\Pi_1 \cup \Pi_2)$ is contained in a unique line that meets Π_1 and Π_2 , giving a rational map $\mathbb{P}^5 \dashrightarrow \Pi_1 \times \Pi_2$. To resolve this rational map, let $\tilde{\mathbb{P}}^5$ be the blow-up of \mathbb{P}^5 along Π_1 and Π_2 , and let p and q be as shown:

$$\begin{array}{ccc} \tilde{\mathbb{P}}^5 & \xrightarrow{p} & \Pi_1 \times \Pi_2 \\ q \downarrow & & \\ \mathbb{P}^5 & & \end{array}$$

Let $F_1, F_2 \subset \tilde{\mathbb{P}}^5$ be the exceptional divisors, let H be the hyperplane class on \mathbb{P}^5 , and let H_1, H_2 denote the pullbacks of the hyperplane classes on $\Pi_1 \times \Pi_2$. We find that $p^*H_i = q^*H - F_i$, so $p^*(H_1 + H_2) = q^*(2H) - F_1 - F_2$; that is, divisors of type $(1, 1)$ on $\Pi_1 \times \Pi_2$ correspond to quadrics in \mathbb{P}^5 containing Π_1 and Π_2 .

By the Bertini theorem and the adjunction formula, a generic intersection of three divisors of type $(1, 1)$ is an elliptic curve $E \subset \Pi_1 \times \Pi_2$. The projection onto either factor maps E isomorphically onto a smooth cubic curve $E_i \subset \Pi_i$, as follows. The three $(1, 1)$ -divisors determine a 3×3 matrix of linear forms on Π_i , and the image of E is the locus where the determinant vanishes. The fibers of E over E_1 are linear spaces in Π_2 , but E contains no lines, so $E \rightarrow E_i$ is one-to-one. Thus $p_g(E_i) = 1 = p_a(E_i)$, so E_i is smooth and $E \rightarrow E_i$ is an isomorphism.

Now p is a \mathbb{P}^1 -bundle, so $\tilde{T} := p^{-1}(E)$ is an elliptic ruled surface; and $F_1, F_2 \subset \tilde{\mathbb{P}}^5$ give sections of p , hence sections of $\tilde{T} \rightarrow E$. Let $T = q(\tilde{T}) \subset \mathbb{P}^5$. From our analysis of the projections $E \rightarrow E_i \subset \Pi_i$, it follows that $\tilde{T} \rightarrow T$ is an isomorphism and $T \cap \Pi_i = E_i$. \square

Proposition 4. *Let $T \subset \mathbb{P}^5$ be an elliptic ruled surface as in Proposition 3. Then the homogeneous ideal of T is generated by quadrics and cubics. Moreover,*

$$\begin{aligned} h^0(\mathcal{I}_T(1)) &= 0 \\ h^0(\mathcal{I}_T(2)) &= 3 \\ h^0(\mathcal{I}_T(3)) &= 20. \end{aligned}$$

Proof. The first statement follows from [Hom80, Thm. 3.3]. The second can be found in [Hom80, §2], or we can prove it as follows. From the inclusion $T \subset (\Pi_1 \cup T \cup \Pi_2) \subset \mathbb{P}^5$ we get an exact sequence

$$0 \rightarrow \mathcal{I}_{(\Pi_1 \cup T \cup \Pi_2)/\mathbb{P}^5} \rightarrow \mathcal{I}_T/\mathbb{P}^5 \rightarrow \mathcal{I}_{T/(\Pi_1 \cup T \cup \Pi_2)} \rightarrow 0.$$

Since $\Pi_1 \cup T \cup \Pi_2$ is a complete intersection of three quadrics, we can compute cohomology of the first term using a Koszul complex. The third term is isomorphic to

$$\mathcal{I}_{E_1/\Pi_1} \oplus \mathcal{I}_{E_2/\Pi_2} = \mathcal{O}_{\Pi_1}(-3) \oplus \mathcal{O}_{\Pi_2}(-3).$$

Then the calculation is straightforward. \square

Proof of Theorem 2. First we claim that the Hilbert scheme of surfaces T of the form appearing in Proposition 3 is smooth and irreducible of dimension 36. Indeed, it is isomorphic to an open subset of a $\text{Gr}(3, 9)$ -bundle over $\text{Gr}(3, 6) \times \text{Gr}(3, 6)$. To see this, observe that each T arises from a *unique* pair of planes Π_1, Π_2 : by intersecting the three quadrics containing T we recover Π_1 and Π_2 .

Next, the Hilbert scheme of pairs $T \subset X$, where X is a smooth cubic fourfold, is smooth and irreducible of dimension 55. It is an open subset of a \mathbb{P}^{19} -bundle over the previous Hilbert scheme. And it is non-empty: we write down an explicit smooth cubic fourfold containing one of our elliptic ruled surfaces in Section 6.

Next, we claim that if X is a smooth cubic fourfold containing T then $X \in \mathcal{C}_{18}$. The intersection form on a smooth cubic fourfold X containing T is as follows:

$$\begin{array}{c|cc} & h^2 & T \\ \hline h^2 & 3 & 6 \\ T & 6 & 18 \end{array}$$

Only $T^2 = 18$ needs to be justified. By [Has00, §4.1], we have

$$T^2 = 6h_T^2 + 3h_T.K_T + K_T^2 - \chi_T,$$

where χ_T is the topological Euler characteristic and h_T is the hyperplane class restricted to T . Because T is an elliptic ruled surface, we have $K_T^2 = 0 = \chi_T$. Moreover we have $K_T = -E_1 - E_2$, so $h_T.K_T = -6$.

Modding out the action of $\mathrm{PGL}(6)$, we get a map from an irreducible, 20-dimensional space of pairs (T, X) to an irreducible 19-dimensional space \mathcal{C}_{18} . To show that this is dominant, it is enough to show that for a generic pair (T, X) , the deformation space $H^0(\mathcal{N}_{T/X})$ is at most 1-dimensional. By semi-continuity it is enough to verify this for one pair. We do this with Macaulay2 in Section 6. \square

Remark 5. The ruled surface $T \subset X$ determines an elliptic curve E in the variety of lines $F_1(X)$. By [Ran95, Cor. 5.1], such a curve moves in a family of dimension at least 1.

2. CONSTRUCTING THE DEL PEZZO FIBRATION

Theorem 6. *Let X be a cubic fourfold containing an elliptic ruled surface T as in Theorem 2, and let*

$$\pi: X' := \mathrm{Bl}_T(X) \rightarrow \mathbb{P}^2$$

be the map induced by the linear system of quadrics containing T . For generic X , the generic fiber of π is a del Pezzo surface of degree 6, and the preimages of the curves $E_1, E_2 \subset T$ induce trisections of π .

Proof. We will show that for a generic choice of two quadrics $Q_1, Q_2 \supset T$ and a cubic $X \supset T$, we have $Q_1 \cap Q_2 \cap X = T \cup S$, where S is a del Pezzo surface of degree 6.

We have seen that any quadric containing T also contains the two planes Π_1, Π_2 appearing in Theorem 2. But by Proposition 4 we know that T is cut out by cubics, so a generic cubic containing T does not contain Π_1 or Π_2 . Thus by the Bertini theorem, we can say that $Q_1 \cap Q_2 \cap X = T \cup S$, where S is irreducible and smooth away from T . First we argue that S is in fact smooth everywhere, and that S and T meet transversely in a smooth curve D .

Let $V = Q_1 \cap Q_2$. We claim that V has ordinary double points

$$s_{11}, s_{12}, s_{13} \in \Pi_1 \cap T, \quad s_{21}, s_{22}, s_{23} \in \Pi_2 \cap T.$$

(Compare [Kap09, Thm. 2.1].) To see this, let

$$\begin{array}{ccc} \tilde{\mathbb{P}}^5 & \xrightarrow{p} & \Pi_1 \times \Pi_2 \\ q \downarrow & & \\ \mathbb{P}^5 & & \end{array}$$

be as in the proof of Proposition 3, and let $W \subset \Pi_1 \times \Pi_2$ be the intersection of two $(1, 1)$ -divisors corresponding to Q_1, Q_2 . (Perhaps by coincidence, W is a sextic del Pezzo surface.) The projections $W \rightarrow \Pi_i$ each contract three lines; thus $V = q(p^{-1}(W))$ is smooth away from six ordinary double points as claimed.

Now consider the blow-up

$$r: \text{Bl}_T(\mathbb{P}^5) \rightarrow \mathbb{P}^5,$$

let $F \subset \text{Bl}_T(\mathbb{P}^5)$ be the exceptional divisor, and let $V' \subset \text{Bl}_T(\mathbb{P}^5)$ be the proper transform of V . This is a small resolution of V ; indeed, it is the flop of $\tilde{V} := p^{-1}(W)$ along the six exceptional lines. Let $T' = V' \cap F$, which is the blow-up of T at the six points s_{ij} .

Let X' be a generic divisor in the linear system $|r^*(3H) - F|$. Because T is cut out by cubics, this linear system is basepoint-free, so X' is smooth, and moreover the surface $S' := X' \cap V'$ and the curve $D' := X' \cap T'$ are smooth. We claim that r maps S' isomorphically onto $S = r(S')$, and D' isomorphically onto $D = r(D')$. For this it is enough to observe that X' meets the exceptional lines of $V' \rightarrow V$ in one point each; otherwise one of the lines would be contained in the base locus of $|r^*(3H) - F|$, which is impossible. (As Q_1 and Q_2 vary, these six points give rise to the two trisections of π .)

Finally, we argue that S is a del Pezzo surface of degree 6. By adjunction we have $K_{S'} = r^*H - F$, so $K_S = H - D$. But $D = S \cap T = S \cap Q_3$, so $K_S = H - 2H = -H$, so S is a del Pezzo surface. And $S \cup T = Q_1 \cap Q_2 \cap X$ has degree 12, and T has degree 6, so S has degree 6. \square

Remark 7. The curve $D = T \cap S$ appearing in the previous proof is smooth of degree 12 and genus 7. To see this, note again that on S we have $D \sim 2H$, so $D.H = 12$ and $\deg K_D = K_S.D + D^2 = 12$.

3. RATIONALITY OF SEXTIC DEL PEZZO SURFACES

To determine when our fibration in sextic del Pezzo surfaces $\pi: X' \rightarrow \mathbb{P}^2$ is rational, we review rationality properties of sextic del Pezzos in general.

Let S be a del Pezzo surface of degree six over a perfect field F . (In our application F is the function field of \mathbb{P}^2 .) Over an algebraic closure \bar{F} , the surface $\bar{S} = S_{\bar{F}}$ is isomorphic to $\mathbb{P}_{\bar{F}}^2$ blown up at three non-collinear points. The lines $((-1)$ -curves) on \bar{S} consist of the exceptional divisors and the proper transforms of lines joining pairs of the points; these form a hexagon with dihedral symmetry $\mathfrak{D}_{12} \simeq \mathfrak{S}_2 \times \mathfrak{S}_3$.

The Galois action on the Picard group gives a representation

$$\rho_S: \text{Gal}(\bar{F}/F) \rightarrow \mathfrak{S}_2 \times \mathfrak{S}_3,$$

where the first factor indexes the geometric realizations of S as a blow-up of \mathbb{P}^2 and the second factor corresponds to the conic bundle structures. Let K/F denote the quadratic étale algebra associated with first factor and L/F a cubic étale algebra associated with the second factor. Blunk [Blu10] has studied Azumaya algebras B/K and Q/L with the following properties:

- The Brauer-Severi variety $\text{BS}(B)$ has dimension two over K , and there is a birational morphism

$$S_K \rightarrow \text{BS}(B)$$

realizing S_K as the blow-up over a cycle of three points;

- $\text{BS}(Q)$ has dimension one over L , and there is a morphism

$$S_L \rightarrow \text{BS}(Q)$$

realizing S_L as a conic fibration with two degenerate fibers;

- the corestrictions $\text{cor}_{K/F}(B)$ and $\text{cor}_{L/F}(Q)$ are split over F ;
- B and Q both contain a copy of the compositum KL , and thus are split over this field.

Proposition 8. *The following are equivalent:*

- (1) S is rational over F ;
- (2) S admits an F -rational point;
- (3) S admits a zero-cycle of degree prime to six;
- (4) the Brauer classes B and Q are trivial.

Proof. The implications (1) \Rightarrow (2) \Rightarrow (3) are clear. The implication (2) \Rightarrow (1) is [Man66, Cor. 1 to Thm. 3.10]. The implication (3) \Rightarrow (4) is straightforward: note that B is trivial if and only if $\text{BS}(B)$ admits a zero-cycle of degree prime to 3, that Q is trivial if and only if $\text{BS}(Q)$

admits a zero-cycle of odd degree, and that S_K maps to $\text{BS}(B)$ and S_L maps to $\text{BS}(Q)$. The implication (4) \Rightarrow (2) is [Blu10, Cor. 3.5]. \square

Remark 9. It may happen that a del Pezzo surface S has maximal Galois representation ρ_S , but B and Q are nonetheless trivial. For example, let $F = \mathbb{C}(t)$, and choose a surjective representation

$$\rho: \text{Gal}(\bar{F}/F) \rightarrow \mathfrak{S}_2 \times \mathfrak{S}_3$$

corresponding to a branched cover $C \rightarrow \mathbb{P}^1$. The geometric automorphism group admits a split exact sequence [Blu10, §2]

$$1 \rightarrow T \rightarrow \text{Aut}(\bar{S}) \rightarrow \mathfrak{S}_2 \times \mathfrak{S}_3 \rightarrow 1.$$

Here T is a torus with a natural $\mathfrak{S}_2 \times \mathfrak{S}_3$ action via conjugation. Composing ρ with the splitting, we obtain a del Pezzo surface $S/\mathbb{C}(t)$ of degree six with $\rho_S = \rho$. However, the Brauer group of any complex curve is trivial, so the Brauer classes above necessarily vanish.

4. RATIONALITY OF OUR CUBIC FOURFOLDS

Proof of Theorem 1. Let $U \subset \mathcal{C}_{18}$ be the Zariski open set parametrizing cubics fourfolds as in Theorem 6. Given $X \in U$, we have the blow-up $r: X' \rightarrow X$ along T and the sextic del Pezzo fibration $\pi: X' \rightarrow \mathbb{P}^2$. Let $S' \subset X'$ be a smooth fiber of π , and let $S = r(S')$. We claim that if there is a cohomology class $\Sigma \in H^4(X, \mathbb{Z}) \cap H^{2,2}(X)$ with $\Sigma.S = 1$, then X is rational. The integral Hodge conjecture holds for cubic fourfolds [Voi13, Thm. 1.4], so we can promote Σ to an algebraic cycle. Then $\Sigma' = r^*\Sigma$ satisfies $\Sigma'.S' = 1$ by the projection formula. Proposition 8 implies that X' is rational over the function field $\mathbb{C}(\mathbb{P}^2)$, hence over \mathbb{C} , so X is rational over \mathbb{C} .

Thus our goal is to produce a countable dense set of divisors

$$\mathcal{C}_K \subset \mathcal{C}_{18}$$

such that if X is in some \mathcal{C}_K then there is a Hodge class Σ with $\Sigma.S = 1$.

Let $h \in H^2(X, \mathbb{Z})$ be the hyperplane class, let L be the lattice underlying $H^4(X, \mathbb{Z})$, and let $L^0 = h^{\perp} \subset L$ be the primitive cohomology, which may be expressed [Has00, Prop. 2.1.2]:

$$L^0 \simeq \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus E_8^{\oplus 2},$$

where E_8 is the positive definite quadratic form associated with the corresponding Dynkin diagram. Note that L^0 is even.

Consider positive definite rank-three overlattices

$$\begin{array}{c|cc} & h^2 & S \\ \hline h^2 & 3 & 6 \\ S & 6 & 18 \end{array} \subset K_{a,b} := \begin{array}{c|ccc} & h^2 & S & \Sigma \\ \hline h^2 & 3 & 6 & a \\ S & 6 & 18 & 1 \\ \Sigma & a & 1 & b \end{array}$$

with discriminant

$$\Delta = -3 + 12a - 18a^2 + 18b.$$

The lattice $h^{2\perp} \subset K_{a,b}$ is even if and only if $a \equiv b \pmod{2}$. We assume this parity condition from now on.

Nikulin's results on embeddings of lattices [Nik79, §1.14] imply that the embedding

$$\langle h^2, S \rangle \hookrightarrow L$$

extends to an embedding of $K_{a,b}$ in L ; compare [Has99, §4]. Replacing Σ with $\Sigma + m(3h^2 - S)$ for a suitable $m \in \mathbb{Z}$, we may assume that $a = -1, 0, 1$. Thus positive integers $\Delta \equiv 9 \pmod{12}$ arise as discriminants, each for precisely one lattice, denoted K_Δ from now on.

Excluding finitely many small Δ , the lattice K_Δ defines a divisor

$$\mathcal{C}_{K_\Delta} \subset \mathcal{C}_{18}.$$

See [Has16, §2.3] for details on which Δ must be excluded, and [AT14, §4] for details of a similar calculation in \mathcal{C}_8 .

The divisors \mathcal{C}_{K_Δ} intersecting U are the ones we want. The density in the Euclidean topology follows from the Torelli theorem [Voi86] and from [Voi07, 5.3.4]. \square

Let X be a cubic fourfold containing an elliptic ruled surface T as in Theorem 2, and let $\pi: X' \rightarrow \mathbb{P}^2$ be the associated fibration into del Pezzo surfaces S of degree 6. A labelling of such a cubic fourfold is a choice of lattice

$$\langle h^2, T \rangle = \langle h^2, S \rangle \subset A(X) := H^4(X, \mathbb{Z}) \cap H^{2,2}(X)$$

associated with such a fibration.

Proposition 10 (Hodge-theoretic interpretation). *Let X be a labelled cubic fourfold of discriminant 18, and let Λ be the Hodge structure on the orthogonal complement of the labelling lattice. Then there exists an embedding of polarized Hodge structures*

$$\Lambda(-1) \hookrightarrow H_{\text{prim}}^2(Y', \mathbb{Z}),$$

where (Y', f') is a polarized K3 surface of degree two, and $\Lambda(-1)$ is an index-three sublattice, expressible as

$$\Lambda(-1) \cong \langle \eta' \rangle^\perp,$$

for some $\eta' \in H^2(Y', \mathbb{Z})$ whose image in $H^2(Y', \mathbb{Z}/3\mathbb{Z})/\langle f' \rangle$ is isotropic under the intersection form modulo 3. If $A(X)$ has rank 2, then the class η' gives rise to a non-trivial element in $\text{Br}(Y')[3]$.

Proof. The discriminant group of the lattice $\Lambda(-1)$ is isomorphic to $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$ [Has00, Prop. 3.2.5]. On the other hand, Theorem 9 in [MSTVA14] shows that there exist polarized K3 surfaces (Y', f') of degree two such that $H^2_{\text{prim}}(Y', \mathbb{Z})$ contains a lattice Γ of index three having the same rank and discriminant group as $\Lambda(-1)$. The lattices $\Lambda(-1)$ and Γ are even, indefinite and have few generators for their discriminant groups relative to their rank (2 vs. 21). Results of Nikulin [Nik79] now imply they are isometric. The isomorphism

$$\begin{aligned} H^2(Y', \mathbb{Z})/\langle f' \rangle \otimes \mathbb{Z}/3\mathbb{Z} &\xrightarrow{\sim} \text{Hom}(\langle f' \rangle^\perp, \mathbb{Z}/3\mathbb{Z}) \\ v \otimes 1 &\mapsto [t \mapsto v \cdot t \bmod 3] \end{aligned}$$

identifies Γ with the kernel of a map $\langle f' \rangle^\perp \rightarrow \mathbb{Z}/3\mathbb{Z}$ corresponding to some $\eta' \otimes 1$, so $\Gamma = \{t \in \langle f' \rangle^\perp : t \cdot \eta' \in 3\mathbb{Z}\}$; see [vG05, §2.1] for details.

If $A(X)$ has rank 2, then $\text{NS}(Y') = \langle f' \rangle$. The exponential sequence then gives rise to the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^2(Y', \mathbb{Z})/\langle f' \rangle & \longrightarrow & H^2(Y', \mathcal{O}_{Y'}) & \longrightarrow & \text{Br}(Y') \longrightarrow 0 \\ & & \downarrow \times 3 & & \downarrow \times 3 & & \downarrow \times 3 \\ 0 & \longrightarrow & H^2(Y', \mathbb{Z})/\langle f' \rangle & \longrightarrow & H^2(Y', \mathcal{O}_{Y'}) & \longrightarrow & \text{Br}(Y') \longrightarrow 0 \end{array}$$

The snake lemma then shows that to $[\eta'] \in H^2(Y', \mathbb{Z}/3\mathbb{Z})/\langle f' \rangle$ there corresponds a class in $\text{Br}(Y')[3]$. \square

Proposition 10 could also have been obtained using the results on Mukai lattices in [Huy15, §2].

5. SINGULAR FIBERS AND DISCRIMINANT CURVES

In this section we study the locus where fibers of $\pi: X' \rightarrow \mathbb{P}^2$ degenerate, and its relation to the Azumaya algebras B/K and Q/L of Section 3.

A *singular del Pezzo surface* is a surface S with ADE singularities and ample anticanonical class. Those of degree six ($K_S^2 = 6$), are classified as follows [CT88, Prop. 8.3]:

- type I: S has one A_1 singularity and is obtained by blowing up \mathbb{P}^2 in three collinear points, and blowing down the proper transform of the line containing them;

- type II: S also has one A_1 singularity and is obtained by blowing up \mathbb{P}^2 in two infinitely near points and a third point not on the line associated with the infinitely near points, then blowing down the proper transform of the first exceptional divisor over the infinitely near points;
- type III: S has two A_1 singularities and is obtained by blowing up two infinitely near points and a third point all contained in a line, and blowing down the proper transforms of the first exceptional divisor over the infinitely near point and the line;
- type IV: S has an A_2 singularity and is obtained by blowing up a curvilinear subscheme of length three not contained in a line and blowing down the first two exceptional divisors;
- type V: S has an A_1 and an A_2 singularity and is obtained by blowing up a curvilinear subscheme contained in a line, and blowing down the proper transforms of the first two exceptional divisors and the line.

Types I and II occur in codimension one and correspond to conjugacy classes of involutions associated with the factors of $\mathfrak{S}_2 \times \mathfrak{S}_3$. Type III occurs in codimension two and corresponds to conjugacy classes of Klein four-groups. Type IV also occurs in codimension two and corresponds to three cycles. Type V occurs in codimension three and corresponds to the full group.

Definition 11. Let P be a smooth complex projective surface. A *good del Pezzo fibration* consists of a smooth fourfold \mathcal{S} and a flat projective morphism $\pi: \mathcal{S} \rightarrow P$ with the following properties:

- The fibers of π are either smooth or singular del sextic del Pezzo surfaces. Let B_I, \dots, B_V denote the closure of the corresponding loci in P .
- B_I is a non-singular curve.
- B_{II} is a curve, non-singular away from B_{IV} .
- B_{III} is finite and coincides with the intersection of B_I and B_{II} , which is transverse.
- B_{IV} is finite, and B_{II} has cusps at B_{IV} .
- B_V is empty.

Remark 12. The point is that the classifying map from P is transverse to each singular stratum of the moduli stack of singular del Pezzo surfaces. For example, the discriminant locus is cuspidal at points with an A_2 singularity. In particular, good del Pezzo fibrations are Zariski open in the moduli space of all del Pezzo fibrations with fixed invariants.

Proposition 13. *Let $\pi: \mathcal{S} \rightarrow P$ be a good del Pezzo fibration. Then Blunk's construction yields:*

- *a non-singular double cover $Y \rightarrow P$ branched along B_I ;*
- *an element $\eta \in \text{Br}(Y)[3]$;*
- *a non-singular degree-three cover $Z \rightarrow P$ branched along B_{II} ;*
- *an element $\zeta \in \text{Br}(Z)[2]$.*

Proof. As before, let K and L be the quadratic and cubic extensions of $\mathbb{C}(P)$ introduced in Section 3. Let Y and Z denote the normalizations of P in the fields (or étale algebras) K and L respectively.

We first address the double cover. Since B_I is smooth, the double cover branched along B_I is also smooth. Consider the base change $\pi_Y: \mathcal{S} \times_P Y \rightarrow Y$, a sextic del Pezzo fibration with singular fibers of types I, II, III, IV and geometric generic fiber \overline{S} . Let G be the Galois group of KL over K . Let $H_1, H_2 \in \text{Pic}(\overline{S})^G$ be the classes corresponding to the birational morphisms

$$\beta_1, \beta_2: \overline{S} \rightarrow \mathbb{P}_K^2.$$

These specialize to Weil divisor classes in each geometric fiber of π_Y . These are Cartier for fibers of types II and IV; H_1 and H_2 are disjoint from the vanishing cycles, reflecting that the resulting divisors are disjoint from the A_1 and A_2 singularities. For fibers of type I, the specializations of H_1 and H_2 coincide and yield smooth Weil divisors containing the A_1 singularity; their order in the local class group is two. For fibers of type III, they contain the A_2 singularity and have order three in the local class group. In each case the resulting curves are parametrized by \mathbb{P}^2 .

We relativize this as follows: Let $\mathcal{H} \rightarrow P$ denote the relative Hilbert scheme parametrizing connected genus zero curves of anticanonical degree three. The analysis above shows that its Stein factorization takes the form

$$\mathcal{H} \rightarrow Y \rightarrow P,$$

where the first morphism is an étale \mathbb{P}^2 -bundle. Thus we obtain the desired class $\eta \in \text{Br}(Y)[3]$.

Next we turn to the triple cover. Let $\mathcal{H}' \rightarrow P$ denote the relative Hilbert scheme parametrizing connected genus zero curves of anticanonical degree two. These are fibers of the conic bundle fibrations

$$\gamma_1, \gamma_2, \gamma_3: \overline{S} \rightarrow \mathbb{P}_L^1.$$

A case-by-case analysis shows the conics in type I-IV fibers are still parametrized by \mathbb{P}^1 's. Repeating the analysis above, the Stein factorization

$$\mathcal{H}' \rightarrow Z \rightarrow P$$

consists of an *étale* \mathbb{P}^1 -bundle followed by a triple cover ramified along B_{II} . The fact that B_{II} has cusps at the points of threefold ramification shows that Z is nonsingular. Indeed, *étale* locally such covers take the form

$$\mathbb{A}^2 = \{(r_1, r_2, r_3) : r_1 + r_2 + r_3 = 0\} \rightarrow \mathbb{A}^2 / \{(12)\} \rightarrow \mathbb{A}^2 / \mathfrak{S}_3$$

where $Z \rightarrow P$ corresponds to the second morphism, branched over the discriminant divisor which is cuspidal at the origin. The *étale* \mathbb{P}^1 bundle yields $\zeta \in \text{Br}(Z)[2]$, the desired Brauer class. \square

Proposition 14. *Let $\pi: \mathcal{S} \rightarrow P$ be a good del Pezzo fibration and fix*

$$\begin{aligned} b_{IV} &= \chi(B_{IV}) = |B_{IV}| \\ b_{III} &= \chi(B_{III}) = |B_{III}| = |B_I \cap B_{II}| \\ b_{II} &= \chi(B_{II} \setminus (B_{III} \cup B_{IV})) \\ b_I &= \chi(B_I \setminus B_{III}), \end{aligned}$$

where χ is the topological Euler characteristic. Then we have

$$\chi(\mathcal{S}) = 6\chi(P) - b_I - b_{II} - 2b_{III} - 2b_{IV}.$$

Proof. This follows from the stratification of the fibration by singularity type. A smooth sextic del Pezzo surface has $\chi = 6$. For types I and II we have $\chi = 5$; for types III and IV we have $\chi = 4$. \square

We specialize Proposition 14 to the case where the base is \mathbb{P}^2 , using Bezout's Theorem and the genus formula:

Corollary 15. *Let $\pi: \mathcal{S} \rightarrow \mathbb{P}^2$ be a good del Pezzo fibration; write $d_I = \deg(B_I)$ and $d_{II} = \deg(B_{II})$. Then we have*

$$\chi(\mathcal{S}) = 14 + (d_I - 1)(d_I - 2) + (d_{II} - 1)(d_{II} - 2) - 3b_{IV}.$$

We specialize further to the del Pezzo fibrations appearing in our main construction:

Proposition 16. *Let $\pi: X' \rightarrow \mathbb{P}^2$ be as in Theorem 6. The discriminant locus contains a sextic curve with nine cusps, projectively dual to a smooth plane cubic.*

Proof. Again let

$$\begin{array}{ccc} \tilde{\mathbb{P}}^5 & \xrightarrow{p} & \Pi_1 \times \Pi_2 \\ q \downarrow & & \\ \mathbb{P}^5 & & \end{array}$$

be as in the proof of Proposition 3, and recall that p is a \mathbb{P}^1 -bundle and that $E \subset \Pi_1 \times \Pi_2$ was obtained as the intersection of three $(1, 1)$ -divisors. View this as three hyperplane sections of $\Pi_1 \times \Pi_2$ embedded in \mathbb{P}^8 ; by projective duality we get a plane Π_3 slicing the determinantal cubic in the dual \mathbb{P}^8 , hence a smooth cubic curve $E_3 \subset \Pi_3$. The map π takes values in the dual plane Π_3^\vee , and we claim that over the dual curve E_3^\vee , the fibers of π are singular.

Indeed, a line $L \subset \Pi_3$ determines a complete intersection of two $(1, 1)$ -divisors $W_L \subset \Pi_1 \times \Pi_2$. By standard arguments, W_L is singular if and only if L is tangent to E_3 , and the singularities of W_L are disjoint from E . Thus $\tilde{V}_L = p^{-1}(W_L)$ is singular along a line disjoint from $\tilde{T} = p^{-1}(E)$, and $V_L = q(\tilde{V}_L)$ is singular along a line that meets $T = q(\tilde{T})$ in only two points: one contained in $E_1 = T \cap \Pi_1$, and one in E_2 . The cubic X meets this line in these two points and at least one more, yielding a singularity of the residual surface S_L , which is the fiber of π over the point $L^\vee \in \Pi_3^\vee$. \square

Remarks 17. (a) The net of $(1, 1)$ -divisors containing E is identified with the net of quadrics containing T , and the discriminant sextic of the latter is the square of the discriminant cubic of the former.

(b) The relation between the elliptic curves $E \subset \Pi_1 \times \Pi_2$ and E_1, E_2, E_3 in Π_1, Π_2, Π_3 is explained in [Ng95] or [BH16, §5.2].

(c) The two trisections of $\pi: X' \rightarrow \mathbb{P}^2$ are also ramified over this cuspidal sextic, hence are identified with the triple cover Z of Proposition 13, and very likely with the elliptic ruled surface T . Clearly there is a lot of interesting geometry to explore here.

Proposition 18. *If the fibration $\pi: X' \rightarrow \mathbb{P}^2$ is good in the sense of Definition 11, then the curve $B_I \subset \mathbb{P}^2$ is a sextic, and the double cover Y is a K3 surface of degree 2.*

Proof. Since $X' = \text{Bl}_T(X)$ and $\chi(T) = 0$, we have

$$\chi(X') = \chi(X) = 1 + 1 + 23 + 1 + 1 = 27.$$

In the previous proposition we have seen that $d_{II} = 6$ and $b_{IV} = 9$. Thus from Corollary 15 we find that $d_I = 6$. For the last statement, recall that Y is the double cover of \mathbb{P}^2 branched over B_I . \square

Remark 19. Let f denote the degree-two polarization on Y associated with the double cover. We expect that (Y, f, η) coincides (up to sign) with the triple (Y', f', η') obtained via Hodge theory in Proposition 10.

In the next section we will see that there are cubics $X \in \mathcal{C}_{18}$ for which $\pi: X' \rightarrow \mathbb{P}^2$ is good.

6. AN EXPLICIT EXAMPLE

The computations below were verified symbolically with Macaulay2 [GS]. Code is available on the arXiv as an ancillary file.

Let $\mathbb{P}^5 = \text{Proj}(\mathbb{F}_5[x_0, \dots, x_5])$. Define quadrics

$$Q_1 = 3x_0x_3 + 2x_0x_5 + 4x_1x_3 + 2x_1x_4 + x_2x_3 + x_2x_4 + 2x_2x_5;$$

$$Q_2 = x_0x_3 + 2x_0x_5 + x_1x_3 + 3x_1x_5 + 2x_2x_4 + 3x_2x_5;$$

$$Q_3 = 2x_0x_4 + x_0x_5 + x_1x_3 + 2x_1x_5 + 4x_2x_3 + 3x_2x_5.$$

Each quadric contains the planes

$$\{x_0 = x_1 = x_2 = 0\} \quad \text{and} \quad \{x_3 = x_4 = x_5 = 0\}.$$

The sextic elliptic ruled surface T , obtained by saturating the ideal generated by the three quadrics with respect to the defining ideals of the planes, is cut out by Q_1, Q_2, Q_3 , and the two cubics

$$\begin{aligned} x_3^3 + 2x_3x_4^2 + x_3x_4x_5 + 4x_3x_5^2 + 4x_4^3 + 4x_5^3 &= 0, \\ x_0^3 + 4x_0^2x_1 + x_0^2x_2 + 2x_0x_1^2 + 2x_0x_1x_2 + 4x_0x_2^2 + x_1^3 + 3x_1^2x_2 + x_2^3 &= 0. \end{aligned}$$

The surface T is contained in the cubic fourfold X cut out by

$$\begin{aligned} f := & x_0^3 + 4x_0^2x_1 + x_0^2x_2 + x_0^2x_3 + 3x_0^2x_4 + 3x_0^2x_5 + 2x_0x_1^2 + 2x_0x_1x_2 \\ & + 4x_0x_1x_3 + 3x_0x_1x_4 + 4x_0x_1x_5 + 4x_0x_2^2 + x_0x_2x_3 + 3x_0x_2x_4 \\ & + 2x_0x_2x_5 + 3x_0x_3^2 + 4x_0x_3x_5 + 4x_0x_4^2 + 2x_0x_4x_5 + x_0x_5^2 + x_1^3 \\ & + 3x_1^2x_2 + 4x_1^2x_3 + x_1x_2x_3 + 3x_1x_2x_4 + 4x_1x_2x_5 + 3x_1x_3^2 \\ & + x_1x_3x_4 + 2x_1x_4^2 + x_1x_4x_5 + 2x_1x_5^2 + x_2^3 + 4x_2^2x_3 + x_2^2x_4 \\ & + 4x_2^2x_5 + 4x_2x_3^2 + 3x_2x_3x_5 + 3x_2x_4^2 + 2x_2x_4x_5 + 4x_2x_5^2 + 4x_3^3 \\ & + 3x_3x_4^2 + 4x_3x_4x_5 + x_3x_5^2 + x_4^3 + x_5^3 \end{aligned}$$

A direct computation of the partial derivatives of f shows that X is smooth. The first order deformations of T as a subscheme of X are given by

$$\Gamma(T, \mathcal{N}_{T/X}) = \text{Hom}(\mathcal{I}_T, \mathcal{O}_T).$$

Macaulay2 verifies that this is one-dimensional, as was required at the end of the proof of Theorem 2.

The discriminant locus of the map $\pi : X' := \text{Bl}_T(X) \rightarrow \mathbb{P}^2 = |\mathcal{I}_T(2)|^\vee$ is a reducible curve of degree 12, with two irreducible components:

$$\begin{aligned} B_I : & x^6 + 2x^4y^2 + x^3y^3 + 4x^3y^2z + 2x^3z^3 + 4x^2y^4 + 4x^2y^2z^2 \\ & + 4x^2yz^3 + 4xy^5 + xy^4z + xy^2z^3 + xyz^4 + 2xz^5 + 4y^6 \\ & + 3y^5z + y^3z^3 + y^2z^4 + 4yz^5 = 0, \end{aligned}$$

$$\begin{aligned} B_{II} : & x^6 + 2x^5y + 2x^4y^2 + x^4yz + 4x^3y^3 + 3x^3y^2z + 4x^3yz^2 + x^3z^3 \\ & + 3x^2y^4 + 4x^2y^2z^2 + x^2yz^3 + 3x^2z^4 + 3xy^5 + 2xy^4z \\ & + 3xy^3z^2 + 3xyz^4 + xz^5 + y^5z + 4y^4z^2 + 3y^3z^3 \\ & + 2y^2z^4 + 4yz^5 = 0 \end{aligned}$$

The curve B_I is smooth, and B_{II} has 9 cusps. Their intersection is a reduced 0-dimensional scheme of degree 36 and is thus transverse. Thus π is good in the sense of Definition 11.

Since the relevant Hilbert schemes are smooth and rational, the equations we write down readily lift to characteristic zero. The properties we stipulate are open and thus hold for any such lift.

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